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Stability and Convergence of Nonconforming *hp* Finite-Element Methods

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Abstract—The stability and convergence of nonconforming *hp* finite-element methods, in particular, the mortar finite-element method and its variants, are established based on a new stability measure for these methods. Using a generalized eigenvalue analysis, estimates for this measure are computed numerically. Our numerical results demonstrate that these nonconforming methods prove to be good candidates for *hp* implementation and also behave as well as conforming finite-element methods. The discussion here is primarily in two dimensions, but some extensions to three dimensions are presented as well.

Keywords—*hp* version, Finite elements, Mortar elements, Nonconforming, Stability measure.

1. INTRODUCTION

finite-element analysis often requires assembly of incompatible sub-discretizations during design. For instance, connecting large substructures such as wing and fuselage structures that may have been modeled by different analysts in different groups or organizations is often quite challenging because the finite-element nodes of each component at the common interface are not, in general, coincident. Nonconforming finite-element methods such as the mortar methods have been shown to help in this regard. These techniques allow us to model the domain as being split into several subdomains, each of which is meshed independently. This approach allows mesh refinement to be imposed selectively on those subdomains where it is needed (such as those around corners or other features) which contribute most to the error. Note that there are other useful applications as well for such nonconforming methods. For instance, they can help analyze the interaction of different mathematical problems describing different processes (e.g., a fluid/structure interaction), using appropriate interface coupling conditions. Also, one can allow different discretization methods in different parts of the domain (for example, one may want to combine finite elements with spectral methods or finite elements with boundary elements or finite collocation methods with finite differences, etc.).

After the introduction of the mortar element method [1], there has been a lot of research in its numerical, computational, and implementational aspects [2–4]. These methods are becoming

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increasingly popular as specialized domain decomposition techniques and have been shown to be theoretically stable and optimal even when highly nonquasiuniform meshes are used to capture singularities [4]. It has also been well established [1,4–6] that the optimal rates afforded by the conforming h, p, and hp discretizations are preserved when such nonconforming methods are used, both in the presence of highly nonquasiuniform meshes and varying polynomial degree. Also, the three-dimensional mortar finite-element method has been analyzed [7,8].

One can find other examples of nonconforming methods in the literature (some defined only at the interelement, rather than the intersubdomain level) [9–11]. In addition, one can also find other specific formulations such as the mortar finite volume methods [12] and mortar multigrid techniques [13,14].

The goal in this paper is to introduce the notion of a *stability measure* which will help characterize the stability and convergence properties of nonconforming methods in an hp framework. We estimate this measure both theoretically and computationally and then extend these ideas to three dimensions. The outline of this work is as follows. We first review the nonconforming mortar (M0) finite-element method and its variant (M1) for a model problem in two dimensions. We present an abstract error estimate result for the nonconforming finite-element solution in Section 2. This proof requires two main ingredients—the *stability measure* and the *extension operator bound*. In Section 3, we estimate the p-dependence of the stability measure for the mortar methods theoretically. We computationally verify these results via a generalized eigenvalue analysis in Section 4. We also show that the stability measure remains independent of the mesh even when highly nonquasiuniform meshes are used. In Section 5, we formulate our model problem as a mixed method which is convenient for implementation. Section 6 contains the hp computations for a Neumann problem with discontinuous coefficients. Finally, in Section 7, we extend our discussion to three dimensions.

Let us consider the following model problem:

$$-\Delta u = f, \quad \text{on } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega_D, \qquad \frac{\partial u}{\partial n} = g, \quad \text{on } \partial\Omega_N, \tag{1}$$

where Ω is a connected bounded polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}$ (where $\partial\Omega_D$ is the Dirichlet boundary and $\partial\Omega_N = \partial\Omega \setminus \partial\Omega_D$ is the Neumann boundary). Using standard Sobolev space notation, we define $H_D^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \partial\Omega_D\}$. The weak variational form of (1) then becomes the following. Find $u \in H_D^1(\Omega)$ satisfying, for all $v \in H_D^1(\Omega)$,

$$a(u,v) \stackrel{\text{def}}{=} \int_\Omega \nabla u . \nabla v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega_N} g v \, ds \stackrel{\text{def}}{=} F(v). \tag{2}$$

This problem has a unique solution. To discretize (2) by the finite-element method, we choose a finite-dimensional space $V_N \subset H_D^1(\Omega)$ of piecewise polynomials and find a solution $u_N \in V_N$ such that $a(u_N, v_N) = F(v_N)$, $\forall\, v_N \in V_N$. This is the *standard conforming* finite-element method.

Let us now partition the domain Ω into S nonoverlapping polygonal subdomains $\{\Omega_i\}_{i=1}^S$, which can be geometrically conforming or nonconforming. By geometrically conforming we mean that $\partial\Omega_i \cap \partial\Omega_j$ $(i < j)$ is either empty, a vertex, or a collection of entire edges of Ω_i and Ω_j. In the latter case, we denote this interface as Γ_{ij} $(i < j)$ and this will consist of individual common edges γ, $\gamma \subset \Gamma_{ij}$. The above conformity condition can be relaxed, since following the arguments of Section 3 in [2], our results extend to nonconforming decompositions as well. Let us define the interface set Γ to be the union of the interface intersections $\partial\Omega_i \cap \partial\Omega_j$ $(i < j)$, which result in a nonempty Γ_{ij}. We further subdivide Ω_i into triangles and parallelograms by *regular* [15] families of meshes $\{\mathcal{T}_h^i\}$. It should be noted that the triangulations over different Ω_i are independent of each other, with no compatibility enforced across interfaces. Hence, it is difficult to define a conforming finite-element space $V_N \subset H_D^1(\Omega)$, since such a (polynomial) space must contain continuous functions.

For $K \subset \mathbb{R}^n$ and $k \geq 0$ integer, let $\mathcal{P}_k(K)$ ($\mathcal{Q}_k(K)$) denote the set of polynomials of total degree (degree in each variable) $\leq k$ on K. Let $\mathbf{k}$ be a degree vector, $\mathbf{k} = \{k_1, k_2, \ldots, k_S\}$, which specifies the degree used over each subdomain and denote $k = \min_{1\leq i\leq S}\{k_i\}$. We assume then that the following families $\{V^i_{h,k_i}\}$ of piecewise polynomial spaces are given on Ω_i:

$$V^i_{h,k_i} = \left\{u \in H^1(\Omega_i) \mid u|_K \in \mathcal{S}_{k_i}(K) \text{ for } K \in \mathcal{T}^i_h,\ u = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_D\right\}, \tag{3}$$

where $\mathcal{S}_k(K)$ is $\mathcal{P}_k(K)$ for K a triangle and $\mathcal{Q}_k(K)$ for K a parallelogram.

We now define the space $\tilde{V}_{h,\mathbf{k}}$ by $\tilde{V}_{h,\mathbf{k}} = \{u \in L_2(\Omega) \mid u|_{\Omega_i} \in V^i_{h,k_i}\}$, a nonconforming space. Note that $\tilde{V}_{h,\mathbf{k}} \not\subset H^1_D(\Omega)$, and hence, cannot be used for finite-element calculations. So, we use, instead, a subspace of $\tilde{V}_{h,\mathbf{k}}$, denoted by $V_{h,\mathbf{k}}$ (defined ahead), which enforces the interdomain continuity in a *weak* sense. In addition to the meshes, the polynomial degrees may also be different across interfaces. We will define a discrete norm to be $\|u\|^2_{1,S} = \sum_{i=1}^{S} \|u\|^2_{H^1(\Omega_i)}$, over $\tilde{V}_{h,\mathbf{k}} + H^1(\Omega)$ which is equivalent to the $H^1(\Omega)$ norm for $u \in H^1(\Omega)$.

Since the meshes $\mathcal{T}^i_h$ are not assumed to conform across interfaces, two separate trace meshes can be defined on Γ_{ij}, one from Ω_i and the other from Ω_j. Given $u \in \tilde{V}_{h,\mathbf{k}}$, we denote the traces of u on Γ_{ij} from each of the domains Ω_i and Ω_j by u^i and u^j, respectively. Then we define

$$V_{h,\mathbf{k}} = \left\{u \in \tilde{V}_{h,\mathbf{k}} \mid \int_\gamma \left(u^i - u^j\right)\ \chi\, ds = 0,\ \forall \chi \in S^{\gamma,ij}_{h,\mathbf{k}},\ \forall \gamma \subset \Gamma_{ij} \subset \Gamma\right\}, \tag{4}$$

where $S^{\gamma,ij}_{h,\mathbf{k}}$ is a space of Lagrange multipliers for each edge $\gamma \subset \Gamma_{ij}$. The discretization to (2) is then given by: find $u_{h,\mathbf{k}} \in V_{h,\mathbf{k}}$ satisfying, for all $v \in V_{h,\mathbf{k}}$,

$$a_S(u_{h,\mathbf{k}}, v) \stackrel{\text{def}}{=} \sum_{i=1}^{S} \int_{\Omega_i} \nabla u_{h,\mathbf{k}} \cdot \nabla v\, dx = F(v). \tag{5}$$

We then define the *energy norm* to be $|u|_{1,S} = (a_S(u,u))^{1/2}$. The following theorem can be proved to be satisfied for Lagrange multiplier spaces in practice [8].

THEOREM 1.1. *Problem (5) has a unique solution provided the spaces $S^{\gamma,ij}_{h,\mathbf{k}}$ contain the constant function.*

Let the mesh $\mathcal{T}^i_h$ induce a mesh $\mathcal{T}^i_h(\Gamma_{ij})$ on Γ_{ij}. Let $\gamma \subset \Gamma_{ij}$ and denote the subintervals of this mesh on γ by I_l, $0 \leq l \leq N$. In the mortar element method, the Lagrange multiplier space is defined to be

$$S^{\gamma,ij}_{h,\mathbf{k}} = \{\chi \in C(\gamma) \mid \chi|_{I_l} \in \mathcal{P}_{k_i}(I_l),\ l = 1, \ldots, N-1,\ \chi|_{I_l} \in \mathcal{P}_{k_i-1}(I_l)\ l = 0, N\}.$$

Note that imposing the mesh and degree on $S^{\gamma,ij}_{h,\mathbf{k}}$ from the domain Ω_i as has been done here is quite arbitrary, and these can be taken from the domain Ω_j as well, without changing the results obtained. The M1 method [5] was defined by taking

$$S^{\gamma,ij}_{h,\mathbf{k}} = \{\chi \in C(\Gamma_{ij}) \mid \chi|_{I_l} \in \mathcal{P}_{k_i-1}(I_l),\ l = 0, \ldots, N\}.$$

Let us denote the space (4), corresponding to the method Mr ($r = 0, 1$) by $(V^r_{h,\mathbf{k}}, S^r_{h,\mathbf{k}})$. Then we see immediately that

$$S^1_{h,\mathbf{k}} \subset S^0_{h,\mathbf{k}}, \tag{6}$$

which implies that the functions in $V^0_{h,\mathbf{k}}$ are more constrained than those in $V^1_{h,\mathbf{k}}$, so that by (4), $V^1_{h,\mathbf{k}} \supset V^0_{h,\mathbf{k}}$. We then have the following theorem [5,8].

THEOREM 1.2. *Problem (5) has a unique solution.*

2. ABSTRACT ERROR ESTIMATES

We will now develop an abstract theory to obtain convergence estimates for the finite-element solution which will help us evaluate the performance of the nonconforming method. Our estimates will be based on two main ingredients: a *stability measure* and an *extension operator*.

2.1. The Stability Measure

Let γ be an edge in $\Gamma_{ij} \subset \Gamma$. Let $\overset{0}{\mathrm{V}}{}^{\gamma}_{h,\mathbf{k}}$ denote the trace space of functions in V^i_{h,k_i} (defined in (3)) that vanish at the end points of γ; i.e.,

$$\overset{0}{\mathrm{V}}{}^{\gamma}_{h,\mathbf{k}} = \left\{ u|_{\gamma},\ u \in V^i_{h,k_i} \right\} \cap H^1_0(\gamma). \tag{7}$$

For any function $z \in L_2(\gamma)$, we then define the space $\mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)$ by

$$\mathcal{L}^{\gamma}_{h,\mathbf{k}}(z) = \left\{ w \in \overset{0}{\mathrm{V}}{}^{\gamma}_{h,\mathbf{k}},\ \int_{\gamma} (w - z)\chi\, ds = 0,\ \forall \chi \in S^{\gamma}_{h,\mathbf{k}} \right\} \tag{8}$$

(where $S^{\gamma}_{h,\mathbf{k}} = S^{\gamma,ij}_{h,\mathbf{k}}$); i.e., $\mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)$ contains all functions $w \in \overset{0}{\mathrm{V}}{}^{\gamma}_{h,\mathbf{k}}$ that satisfy the mortar continuity condition in (4) with respect to z across the interface γ.

We will make the following assumption.

CONDITION B. *For all $\gamma \subset \Gamma_{ij} \subset \Gamma$ and for all $z \in L_2(\gamma)$, $\mathcal{L}^{\gamma}_{h,\mathbf{k}}(z) \neq \emptyset$.*

This is needed to ensure that the space $V_{h,\mathbf{k}}$ has sufficiently many functions. It is equivalent to saying that the matrix B defined later in (28) has rank equal to the number of rows. It is also sufficient for the mixed method formulation of our problem to have a unique solution (see Theorem 5.2).

For M0, we note that $\dim(\overset{0}{\mathrm{V}}{}^{\gamma}_{h,\mathbf{k}}) = \dim(S^{\gamma}_{h,\mathbf{k}})$, and we can solve condition in (8) *uniquely* for w, i.e., $\mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)$ will consist of a *single* element, which we denote by $w = \Pi^{\gamma}_{h,\mathbf{k}}\, z$. Here, $\Pi^{\gamma}_{h,\mathbf{k}}$ is a projection from $L_2(\gamma)$ onto $\overset{0}{\mathrm{V}}{}^{\gamma}_{h,\mathbf{k}}$ defined by

$$\int_{\gamma} \left(\Pi^{\gamma}_{h,\mathbf{k}} z - z \right) \chi\, ds = 0, \qquad \forall \chi \in S^{\gamma}_{h,\mathbf{k}}. \tag{9}$$

(For M1, $\mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)$ will have more than one element.)

We now define stability constants $\alpha^{t,\gamma}_{h,\mathbf{k}}$, $0 \le t \le 1$, as follows:

$$\alpha^{t,\gamma}_{h,\mathbf{k}} = \sup_{\substack{z \in H^t_0(\gamma) \\ z \neq 0}} \ \inf_{w \in \mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)} \frac{\|w\|_{H^t_0(\gamma)}}{\|z\|_{H^t_0(\gamma)}}. \tag{10}$$

For M0, these constants provide upper bounds for the operator norm of $\Pi^{\gamma}_{h,\mathbf{k}}$, considered as a projection from $H^t_0(\gamma)$ ($H^{1/2}_{00}(\gamma)$ for $t = 1/2$) into itself. Finally, we define the *stability measure* (corresponding to t) of the method to be

$$\alpha^t_{h,\mathbf{k}} = \max_{\gamma \subset \Gamma_{ij} \subset \Gamma} \alpha^{t,\gamma}_{h,\mathbf{k}}, \qquad 0 \le t \le 1. \tag{11}$$

REMARK 2.1. The stability measures for $t = 0$, $t = 1$ are easy to calculate. In fact, we could use the equivalent definitions

$$\alpha^{0,\gamma}_{h,\mathbf{k}} = \sup_{\substack{z \in L_2(\gamma) \\ z \neq 0}} \ \inf_{w \in \mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)} \frac{\|w\|_{0,\gamma}}{\|z\|_{0,\gamma}}, \tag{12}$$

$$\alpha^{1,\gamma}_{h,\mathbf{k}} = \sup_{\substack{z \in H^1_0(\gamma) \\ z \neq 0}} \ \inf_{w \in \mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)} \frac{\|w'\|_{0,\gamma}}{\|z'\|_{0,\gamma}}. \tag{13}$$

Then for M0, it is easily seen that if we define

$$\tilde{\alpha}_{h,\mathbf{k}}^{t,\gamma} = \left(\alpha_{h,\mathbf{k}}^{0,\gamma}\right)^{1-t} \left(\alpha_{h,\mathbf{k}}^{1,\gamma}\right)^{t}, \qquad 0 < t < 1, \tag{14}$$

then we have

$$\alpha_{h,\mathbf{k}}^{t,\gamma} \leq \tilde{\alpha}_{h,\mathbf{k}}^{t,\gamma}. \tag{15}$$

In Section 4, we will use (12)–(14) to estimate $\alpha_{h,\mathbf{k}}^{1/2}$ computationally for M0 (the measure for $t = 1/2$ will be the one of greatest interest, as we shall see). For M1, although (15) may not hold, it will still be useful to estimate $\alpha_{h,\mathbf{k}}^{t,\gamma}$ ($t = 0, 1$) and $\tilde{\alpha}_{h,\mathbf{k}}^{1/2,\gamma}$ computationally using (12)–(14).

REMARK 2.2. Suppose $\mathcal{L}_{h,\mathbf{k}}^{\gamma}(z)$ consists of more than one element, as for M1. Then, by imposing further restrictions, we could define not one, but several projections $\Pi_{h,\mathbf{k}}^{\gamma}$. For instance, for M1, we could define $\Pi_{h,\mathbf{k}}^{\gamma} z$ to be the unique element in $\mathcal{L}_{h,\mathbf{k}}^{\gamma}(z)$ that is the mortar (M0) projection. Alternatively, we could define it to be the unique element in $\mathcal{L}_{h,\mathbf{k}}^{\gamma}(z)$ that is minimized in the $H_0^t(\gamma)$ norm (e.g., $t = 0$, 1, or 1/2). Once such a $\Pi = \Pi_{h,\mathbf{k}}^{\gamma}$ is fixed, we could redefine (12),(13) by

$$\alpha_{h,\mathbf{k}}^{0,\gamma} = \sup_{\substack{z \in L_2(\gamma) \\ z \neq 0}} \frac{\|\Pi z\|_{0,\gamma}}{\|z\|_{0,\gamma}}, \qquad \alpha_{h,\mathbf{k}}^{1,\gamma} = \sup_{\substack{z \in H_0^1(\gamma) \\ z \neq 0}} \frac{\|(\Pi z)'\|_{0,\gamma}}{\|z'\|_{0,\gamma}}.$$

Then defining $\tilde{\alpha}_{h,\mathbf{k}}^{t,\gamma}$ by (14), we see that (15) will hold for *all* methods, so that the stability can once again be truly estimated computationally.

2.2. The Extension Operator

In addition to the stability measure, another constant that will appear in our error estimates is a bound for an *extension operator*. We assume that for each $\gamma \subset \Gamma_{ij} \subset \partial\Omega_i$, there exists an extension operator $R_{h,\mathbf{k}}^{\gamma} : \overset{0}{V}{}_{h,\mathbf{k}}^{\gamma} \to V_{h,k_i}^{i}$, satisfying, for all $z \in \overset{0}{V}{}_{h,\mathbf{k}}^{\gamma}$ and any $\epsilon > 0$,

$$R_{h,\mathbf{k}}^{\gamma} z = z, \quad \text{on } \gamma, \qquad R_{h,\mathbf{k}}^{\gamma} z = 0, \quad \text{on } \partial\Omega_i \setminus \gamma, \tag{16}$$

$$\left\| R_{h,\mathbf{k}}^{\gamma} z \right\|_{1,\Omega_i} \leq \beta_{h,\mathbf{k}}^{\epsilon,\gamma} \|z\|_{1/2+\epsilon,\gamma}, \tag{17}$$

with $\beta_{h,\mathbf{k}}^{\epsilon,\gamma}$ a constant independent of z but depending on h, $\mathbf{k}$, and ϵ. As we shall see, in some cases (17) will hold with $\epsilon = 0$ as well.

We define $\beta_{h,\mathbf{k}}^{\epsilon} = \max_{\gamma \subset \Gamma_{ij} \subset \Gamma} \beta_{h,\mathbf{k}}^{\epsilon,\gamma}$. The operator $R_{h,\mathbf{k}}^{\gamma}$ extends a trace polynomial on γ to a piecewise polynomial on Ω_i in a bounded fashion and could be thought of as a finite-dimensional version of, e.g., the inverse Laplacian operator.

2.3. Convergence Estimates

For any $u \in H_D^1(\Omega)$, let us define the set $\check{V}_{h,\mathbf{k}}(u)$ by

$$\check{V}_{h,\mathbf{k}}(u) = \left\{ v \in \tilde{V}_{h,\mathbf{k}} \text{ such that } v(N_l) = u(N_l) \text{ for all } l \right\}, \tag{18}$$

where $\{N_l\}$ denotes the set of all end points of the segments $\gamma \subset \Gamma_{ij} \subset \Gamma$. We have the following theorem. The proof is adapted from [1,4] and is quite general.

THEOREM 2.1. *For any $u \in H_D^1(\Omega)$, let $\{\check{V}_{h,\mathbf{k}}(u)\}$ be defined as in (18). Then there exists a constant C, independent of u, h, and k, such that, for any $\epsilon > 0$,*

$$\|u - u_{h,\mathbf{k}}\|_{1,S} \le C \sum_\gamma \inf_{\mu \in S^\gamma_{h,\mathbf{k}}} \left\| \frac{\partial u}{\partial n} - \mu \right\|_{(H^{1/2}(\gamma))'}$$
$$+ C \inf_{v \in \check{V}_{h,\mathbf{k}}(u)} \left\{ \|u - v\|_{1,S} + \beta^\epsilon_{h,\mathbf{k}} \alpha^{1/2+\epsilon}_{h,\mathbf{k}} \sum_{\gamma \subset \Gamma_{ij} \subset \Gamma} \left(\left\|u - v^i\right\|_{1/2+\epsilon,\gamma} + \left\|u - v^j\right\|_{1/2+\epsilon,\gamma} \right) \right\},$$

where $u_{h,\mathbf{k}}$ is the solution of (5). Moreover, if $\beta^0_{h,\mathbf{k}}$ is finite, we can take $\epsilon = 0$ by replacing $\|\cdot\|_{1/2+\epsilon,\gamma}$ by $\|\cdot\|_{H^{1/2}_{00}(\gamma)}$.

PROOF. We use the second Strang lemma [15], which bounds the error for nonconforming methods in terms of an *approximation* error $(e_A(u))$ and a *consistency* error $(e_C(u))$,

$$\begin{aligned} \|u - u_{h,\mathbf{k}}\|_{1,S} &\le C \left(\inf_{y \in V_{h,\mathbf{k}}} \|u - y\|_{1,S} + \sup_{w \in V_{h,\mathbf{k}}} \frac{|a_S(u,w) - F(w)|}{\|w\|_{1,S}} \right) \\ &= C\left(e_A(u) + e_C(u)\right). \end{aligned} \tag{19}$$

Let us now obtain bounds on each of these errors. For the approximation error term in (19), suppose $\tilde{u}$ is any function in $\tilde{V}_{h,\mathbf{k}}$ matching u at the vertices of each Ω_i; i.e., $\tilde{u} \in \check{V}_{h,\mathbf{k}}(u)$. Let $\gamma \subset \Gamma_{ij} \subset \Gamma$; then the jump $\tilde{u}^i - \tilde{u}^j$ vanishes at the end points of γ. Let $z_\gamma \in \mathcal{L}^\gamma_{h,\mathbf{k}}(\tilde{u}^i - \tilde{u}^j) \in \overset{0}{V}{}^\gamma_{h,\mathbf{k}}$. Then from (8) it follows that

$$\int_\gamma \left(\tilde{u}^i - \left(\tilde{u}^j + z_\gamma\right)\right) \chi \, ds = 0, \qquad \forall \chi \in S^\gamma_{h,\mathbf{k}}. \tag{20}$$

We can now extend z_γ into Ω_i by the extension operator $R^\gamma_{h,\mathbf{k}}$ to obtain $w_\gamma = R^\gamma_{h,\mathbf{k}}(z_\gamma) \in V^i_{h,k_i}$. We note that w_γ vanishes on $\partial\Omega_i \setminus \gamma$ and extend it by zero to Ω. Then, defining $y = \tilde{u} + \sum_\gamma w_\gamma$, we have $y \in V_{h,\mathbf{k}}$. Using (17), (20), and (10), one can then show

$$\begin{aligned} e_A(u) \le \inf_{\tilde{u} \in \check{V}_{h,\mathbf{k}}(u)} &\left\{ \|u - \tilde{u}\|_{1,S} \right. \\ &\left. + \beta^\epsilon_{h,\mathbf{k}} \alpha^{1/2+\epsilon}_{h,\mathbf{k}} \sum_{\gamma \subset \Gamma_{ij} \subset \Gamma} \left(\left\|u - \tilde{u}^i\right\|_{1/2+\epsilon,\gamma} + \left\|u - \tilde{u}^j\right\|_{1/2+\epsilon,\gamma} \right) \right\}. \end{aligned} \tag{21}$$

Next, we consider the consistency error $e_C(u)$ in (19). Let $[w] = (w^i - w^j)$ be the jump of w across $\gamma \subset \Gamma_{ij} \subset \Gamma$. Then using (4), for $\mu \in S^\gamma_{h,\mathbf{k}}$, the error reduces to

$$e_C(u) = \sup_{w \in V_{h,\mathbf{k}}} \frac{\sum_\gamma \int_\gamma [w] \left(\frac{\partial u}{\partial n} - \mu \right) ds}{\|w\|_{1,S}} \le C \sum_\gamma \inf_{\mu \in S^\gamma_{h,\mathbf{k}}} \left\| \frac{\partial u}{\partial n} - \mu \right\|_{(H^{1/2}(\gamma))'}. \tag{22}$$

The proof of the theorem now follows from (19), (21), and (22). ∎

Theorem 2.1, thus, gives us an abstract convergence error estimate on the finite-element solution $u_{h,\mathbf{k}}$ in terms of $\alpha^{1/2+\epsilon}_{h,\mathbf{k}}$ and $\beta^\epsilon_{h,\mathbf{k}}$, which were defined to be the *stability measure* and the bound on the extension operator $R^\gamma_{h,\mathbf{k}}$, respectively. In the following sections, we estimate the dependence of $\alpha^{1/2+\epsilon}_{h,\mathbf{k}}$ on h, k, ϵ for the mortar methods M0 and M1, both theoretically and computationally. For estimating $\beta^\epsilon_{h,\mathbf{k}}$, we have the following theorem.

THEOREM 2.2. *For the spaces $V_{h,\mathbf{k}}$ defined on a regular family of meshes T^i_h, there exists an extension operator $R_{h,\mathbf{k}} : \overset{0}{V}_{h,\mathbf{k}} \to V^i_{h,k_i}$ satisfying (16),(17) with $\beta^\epsilon_{h,\mathbf{k}} = \beta^\epsilon$. Moreover, if h or k is fixed, or for the case of quasiuniform meshes, we may take $\epsilon = 0$.*

The existence of such operators is well known for the p version [16,17], the h version [18,19], the *hp* version with quasiuniform meshes [17], and the *hp* version with general nonquasiuniform meshes [4].

3. THEORETICAL ESTIMATION OF THE STABILITY MEASURE

As pointed out earlier, for M0, the dimensions of $\overset{0}{V}{}^{\gamma}_{h,\mathbf{k}}$ and $S^{\gamma}_{h,\mathbf{k}}$ are equal and the solution to the system of equations obtained from (8) is unique. This defined a unique element by the projection operator $\Pi^{\gamma}_{h,\mathbf{k}}$ in (9) which will be bounded in appropriate operator norms by $\alpha^{t,\gamma}_{h,\mathbf{k}}$. Note, however, that $\dim(\overset{0}{V}{}^{\gamma}_{h,\mathbf{k}}) > \dim(S^{\gamma}_{h,\mathbf{k}})$ for M1 and so we do not have such a projection operator (though, see Remark 2.2).

For the M0 method, let us recall the definition of the projection $\Pi^{\gamma}_{h,\mathbf{k}}$ defined in (9). For $u \in L_2(\gamma)$, $\gamma \subset \Gamma_{ij} \subset \Gamma$, $\Pi^{\gamma}_{h,\mathbf{k}} u = \Pi u \in \overset{0}{V}{}^{\gamma}_{h,\mathbf{k}}$ satisfies

$$\int_{\gamma} \Pi u \chi \, ds = \int_{\gamma} u \chi \, ds, \qquad \forall \chi \in S^{\gamma}_{h,\mathbf{k}}. \tag{23}$$

This projection operator has been shown to be stable in both $L_2(\gamma)$ and $H^1_0(\gamma)$ [4,8,20]. The proof of this stability result, however, required a minor restriction on the spaces $\{V_{h,\mathbf{k}}\}$. This condition essentially says that the mesh refinement cannot be stronger than geometric and it has been shown that meshes commonly used in the h, p, and hp version all satisfy this restriction [8]. Using Theorem 3.1 in [4], we then have the following theorem.

THEOREM 3.1. *For the mortar element methods M0 and M1 we have*

$$\alpha^{1/2+\epsilon}_{h,\mathbf{k}} \leq C k^{3/4+\epsilon/2}. \tag{24}$$

PROOF. We first estimate $\tilde{\alpha}^{1/2+\epsilon,\gamma}_{h,\mathbf{k}}$ by using (14) for M0. Since the constants $\alpha^{0,\gamma}_{h,\mathbf{k}}$ and $\alpha^{1,\gamma}_{h,\mathbf{k}}$ provide upper bounds for the projection operator $\Pi^{\gamma}_{h,\mathbf{k}}$, we have from Theorem 3.1 of [4], $\alpha^{0,\gamma}_{h,\mathbf{k}} = O(k^{1/2})$ and $\alpha^{1,\gamma}_{h,\mathbf{k}} = O(k)$. Using (14) and (15), it is seen that $\alpha^{1/2+\epsilon,\gamma}_{h,\mathbf{k}} \leq C k^{3/4+\epsilon/2}$. This can then be used in conjunction with (11) to prove (24) for the M0 method.

Next, it is clear from (6) and (8) that one can achieve better infimums in definition (10) of the stability measure for the M1 method than the M0 method. Hence, the stability measure $\alpha^{1/2+\epsilon}_{h,\mathbf{k}}$ corresponding to the method M1 also satisfies (24). ∎

4. COMPUTATIONAL ESTIMATION OF THE STABILITY MEASURE

We will now introduce a generalized eigenvalue analysis to compute the stability measures $\alpha^{0,\gamma}_{h,\mathbf{k}}$ and $\alpha^{1,\gamma}_{h,\mathbf{k}}$ for the nonconforming methods M0 and M1, using (12),(13). We will also interpolate between these measures to compute $\tilde{\alpha}^{1/2,\gamma}_{h,\mathbf{k}}$, using (14). For simplicity, we take $\gamma = [-1, 1]$.

4.1. Computational Estimation of $\alpha^{0,\gamma}_{h,\mathbf{k}}$

We use definition (12). This involves functions $z \in L_2(\gamma)$, which may be expanded in the form $z(x) = \sum_{i=1}^{\infty} a_i L_i(x)$, where $L_i(x)$ is the i^{th} Legendre polynomial. Since it is not possible to compute the above sum, we truncate the above series to N_k and then take N_k sufficiently large so that the value of $\alpha^{0,\gamma}_{h,\mathbf{k}}$ remains stable. Therefore, we use

$$z(x) = \sum_{i=1}^{N_k} a_i L_i(x) \tag{25}$$

in (12). Let $\{\psi_i\}_{i=1,\ldots,M}$ be the basis functions for $V_{h,\mathbf{k}}(\gamma)$. We can then write any $w \in V_{h,\mathbf{k}}(\gamma)$ as

$$w(x) = \sum_{j=1}^{M} b_j \psi_j(x). \tag{26}$$

Let $\{\phi_j\}_{j=1,\dots,N}$ be the basis functions for $S_{h,\mathbf{k}}$. Suppose $w \in \mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)$ with z given by (25). Then (8) gives

$$B\vec{b} = C\vec{a}, \tag{27}$$

where B is an $(N \times M)$ matrix and C is an $(N \times N_k)$ matrix given by

$$B_{ij} = \int_{\gamma} \psi_j \phi_i \, dx, \qquad C_{ij} = \int_{\gamma} L_j \phi_i \, dx. \tag{28}$$

Note that if B has rank N, then Condition B is satisfied.

Let us now introduce the matrices D (of size $(M \times M)$) and K (of size $(N_k \times N_k)$) which will be defined as $D_{ij} = \int_{\gamma} \psi_j \psi_i \, dx$ and $K_{ij} = \int_{\gamma} L_j L_i \, dx$, respectively. Note that the matrix D is positive definite, and hence, we can do a Cholesky decomposition $D = L^{\top} L$, where L is a lower triangular $(M \times M)$ matrix. Denoting $A = B\,L^{-1}$ and $\vec{x} = L\,\vec{b}$, equation (27) can be rewritten as the following (generally underdetermined) system of equations:

$$A\vec{x} = C\vec{a}, \tag{29}$$

where A is $(N \times M)$.

Using (25) and (26) it can now be shown that, for any $w \in V_{h,\mathbf{k}}(\gamma)$,

$$\frac{\|w\|^2_{0,\gamma}}{\|z\|^2_{0,\gamma}} = \frac{\vec{x}^{\top}\vec{x}}{\vec{a}^{\top}K\vec{a}}.$$

If $w \in \mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)$, then in the above, we must have $\vec{x}$ satisfying (29). Therefore, taking the infimum over all $w \in \mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)$ gives us

$$\inf_{w \in \mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)} \frac{\|w\|^2_{0,\gamma}}{\|z\|^2_{0,\gamma}} = \min_{\substack{\vec{x} \\ A\vec{x}=C\vec{a}}} \frac{\vec{x}^{\top}\vec{x}}{\vec{a}^{\top}K\vec{a}}. \tag{30}$$

The matrix A is of full rank, and hence, the minimum-norm solution $\vec{x}$ to system in (29) is given by

$$\vec{x} = A^{\top}\left(AA^{\top}\right)^{-1} C\vec{a}. \tag{31}$$

Note that this minimizes $\|w\|_{0,\gamma}$. From (31), the infimum in (30) can be computed to be

$$\inf_{w \in \mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)} \frac{\|w\|^2_{0,\gamma}}{\|z\|^2_{0,\gamma}} = \frac{\vec{a}^{\top}R\vec{a}}{\vec{a}^{\top}K\vec{a}},$$

where $R = C^{\top}(AA^{\top})^{-1}C$. Taking the supremum for all z satisfying (25) will then give us

$$\sup_{z} \inf_{w \in \mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)} \frac{\|w\|^2_{0,\gamma}}{\|z\|^2_{0,\gamma}} = \max_{\vec{a}} \frac{\vec{a}^{\top}R\vec{a}}{\vec{a}^{\top}K\vec{a}}.$$

Hence, our problem of estimating the stability measure $\alpha^{0,\gamma}_{h,\mathbf{k}}$ from (12) is equivalent to solving for the maximum eigenvalue λ, which satisfies the following generalized eigenvalue problem:

$$R\vec{x} = \lambda^2 K\vec{x}. \tag{32}$$

4.2. Computational Estimation of $\alpha_{h,\mathbf{k}}^{1,\gamma}$

We now use (13), and note that any function $z \in H_0^1(\gamma)$ may be expanded as

$$z(x) = \sum_{i=1}^{\infty} a_i \xi_i(x), \tag{33}$$

where $\xi_i(x) = \int_{-1}^{x} L_i(t)\, dt$. As in the previous case, since we cannot compute the infinite sum, we truncate the series to N_k terms and then take N_k sufficiently large for the value of $\alpha_{h,\mathbf{k}}^{1,\gamma}$ to saturate. This gives

$$z(x) = \sum_{i=1}^{N_k} a_i \xi_i(x). \tag{34}$$

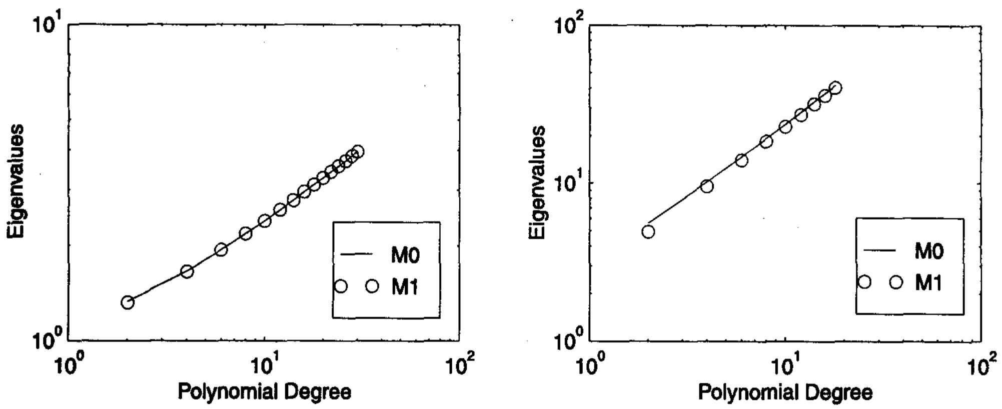
Let $\{\psi_i\}_{i=1,\dots,M}$ and $\{\phi_j\}_{j=1,\dots,N}$ be the basis functions for $V_{h,\mathbf{k}}(\gamma)$ and $S_{h,\mathbf{k}}$, respectively, as before. Let $w \in V_{h,\mathbf{k}}(\gamma)$ defined as (26). Suppose $w \in \mathcal{L}_{h,\mathbf{k}}^{\gamma}(z)$ with z as in (33). Then (8) gives system (27) with $B_{ij} = \int_\gamma \psi_j \phi_i\, dx$ and $C_{ij} = \int_\gamma \xi_j \phi_i\, dx$. The sizes of these matrices B and C are $(N \times M)$ and $(N \times N_k)$, respectively. We also define matrices D and K which are of sizes $(M \times M)$ and $(N_k \times N_k)$, respectively, as $D_{ij} = \int_\gamma \psi_j' \psi_i'\, dx$ and $K_{ij} = \int_\gamma L_j L_i\, dx$. Using these modified definitions of B_{ij}, C_{ij}, D_{ij}, and K_{ij}, we repeat the generalized eigenvalue analysis presented in Section 4.1, which leads us to solve a similar equation as (32) in order to estimate the stability measure $\alpha_{h,\mathbf{k}}^{1,\gamma}$ in (13).

4.3. Results of the Computational Experiments

Since our theory showed that the stability measures are independent of the mesh parameter h, we kept the mesh fixed (taking a uniform mesh with three subintervals) and performed computations to verify the dependence on k. We computed the stability measure constants by estimating the maximum eigenvalue in (32).

Figure 1 shows that the stability constants $\alpha_{h,\mathbf{k}}^{t,\gamma}$ for $t = 0, 1$ grow as $O(k^{1/2})$ and $O(k)$, respectively, for both the methods M0 and M1. We repeated the experiments by varying the number of subintervals to six and nine and observed the same kind of behaviour in the methods. We can now use (14) to bound $\tilde{\alpha}_{h,\mathbf{k}}^{1/2,\gamma}$ by the estimates that we have obtained on $\alpha_{h,\mathbf{k}}^{t,\gamma}$ for $t = 0, 1$ to give

$$\tilde{\alpha}_{h,\mathbf{k}}^{1/2,\gamma} = \left(\alpha_{h,\mathbf{k}}^{0,\gamma}\right)^{1/2} \left(\alpha_{h,\mathbf{k}}^{1,\gamma}\right)^{1/2} = O\left(k^{1/2}\right) O(k) = O\left(k^{3/4}\right). \tag{35}$$



(a) Stability in L_2. (b) Stability in H_1^0.

Figure 1. Stability bounds on $\alpha_{h,\mathbf{k}}^{0,\gamma}$ and $\alpha_{h,\mathbf{k}}^{1,\gamma}$.

REMARK 4.1. The power $k^{3/4}$ in (24) seems to be *sharp*, as has been computationally verified above.

5. MIXED METHOD FORMULATION

It is somewhat cumbersome to implement the nonconforming method (5) due to the constraints $\int_\gamma (u^i - u^j)\chi\, ds = 0$ for all $\chi \in S^{ij}_{h,\mathbf{k}}$ imposed on $V_{h,\mathbf{k}}$. In this section, we rewrite the weak formulation in (5) as a *mixed method* formulation which can be viewed as a convenient method of practically implementing (5). However, these can also be directly implemented [21] as nonconforming methods without the auxiliary variable λ. Also, one can use the method based on *hanging nodes* [22,23].

Multiplying the partial differential equation in (1) by $v \in \tilde{V}_{h,\mathbf{k}}$, integrating by parts, and applying the boundary conditions gives

$$a_S(u,v) + b_S(v,\lambda) = F(v), \tag{36}$$

where $\lambda = -\frac{\partial u}{\partial n}$ is a new unknown variable and $b_S(v,\lambda)$ is the bilinear form defined by

$$b_S(v,\lambda) = \sum_{\gamma\subset\Gamma_{ij}\subset\Gamma} \int_\gamma \left(v^i - v^j\right)\lambda\, ds. \tag{37}$$

We define the space [2], $\tilde{V}_S = \{v \in L^2(\Omega),\ v|_{\Omega_i} \in H^1(\Omega_i),\ v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega_D\}$, equipped with the *energy norm*, and $M = \{\psi \in \mathcal{D}'(\Gamma),\ \psi|_\gamma \in H^{-1/2}(\gamma),\ \forall\gamma \subset \Gamma_{ij} \subset \Gamma\}$ (where $\mathcal{D}'$ is the Schwarz set of distributions) equipped with the norm

$$\|\psi\|_M = \left(\sum_{\gamma\subset\Gamma_{ij}\subset\Gamma} \|\psi\|^2_{H^{-1/2}(\gamma)}\right)^{1/2}. \tag{38}$$

(Here $H^{-1/2}(\gamma)$ stands for the topological dual space of $H^{1/2}(\gamma)$.) Then it can be easily seen that

$$|b_S(v,\psi)| \le C\|v\|_{1,S}\|\psi\|_M,$$

for all $v \in \tilde{V}_S$, $\psi \in M$; i.e., b_S is a bounded bilinear form on $\tilde{V}_S \times M$. Problem (1) can now be stated in *mixed form* [2]: find $(u,\lambda) \in \tilde{V}_S \times M$ such that

$$a_S(u,v) + b_S(v,\lambda) + b_S(u,\chi) = F(v), \tag{39}$$

for all $(v,\chi) \in \tilde{V}_S \times M$.

REMARK 5.1. Note that the linear elasticity equations can also be formulated as a mixed method. The Lagrange multiplier here would be $\vec{\Lambda} = (\Lambda_1, \Lambda_2)$ where $\Lambda_i = \sum_{j=1}^2 \sigma_{ij} n_j$. Here σ_{ij} are the stresses.

Problem (39) has a unique solution [2]. To define an approximation, we choose finite-dimensional subspaces $\tilde{V}_{h,\mathbf{k}} \subset \tilde{V}_S$ and $S_{h,\mathbf{k}} \subset M$, where

$$S_{h,\mathbf{k}} = S_{h,\mathbf{k}}(\Gamma) = \prod_{\gamma\subset\Gamma_{ij}\subset\Gamma} S^\gamma_{h,\mathbf{k}}. \tag{40}$$

We then find $(\tilde{u}_{h,\mathbf{k}}, \lambda_{h,\mathbf{k}}) \in \tilde{V}_{h,\mathbf{k}} \times S_{h,\mathbf{k}}$ satisfying, for all $(v,\lambda) \in \tilde{V}_{h,\mathbf{k}} \times S_{h,\mathbf{k}}$,

$$a_S\left(\tilde{u}_{h,\mathbf{k}}, v\right) + b_S\left(v, \lambda_{h,\mathbf{k}}\right) + b_S\left(\tilde{u}_{h,\mathbf{k}}, \chi\right) = F(v). \tag{41}$$

We have the following theorem.

THEOREM 5.1. *If $u_{h,\mathbf{k}}$ solves (5) and $(\tilde{u}_{h,\mathbf{k}}, \lambda_{h,\mathbf{k}})$ solves (41), then $u_{h,\mathbf{k}} = \tilde{u}_{h,\mathbf{k}}$.*

PROOF. We subtract (5) from (41) and then substitute for $v = \tilde{u}_{h,\mathbf{k}} - u_{h,\mathbf{k}} \in \tilde{V}_{h,\mathbf{k}}$. The theorem then follows by using the coercivity property of the bilinear form $a_S(\cdot,\cdot)$. ∎

Note, however, that to guarantee that the mixed method (41) has a unique solution, the following is a necessary condition:

$$\inf_{\substack{\lambda \in S_{h,\mathbf{k}} \\ \lambda \neq 0}} \sup_{v \in \tilde{V}_{h,\mathbf{k}}} \frac{b_S(v,\lambda)}{\|v\|_{1,S}\|\lambda\|_M} > 0. \tag{42}$$

THEOREM 5.2. *Let Condition B hold; i.e., each $\mathcal{L}^{\gamma}_{h,\mathbf{k}}(z)$ as defined in (8) is nonempty. Then problem (41) has a unique solution.*

PROOF. Given $\lambda \neq 0$ in $S^{\gamma}_{h,\mathbf{k}}$, let $\gamma \subset \Gamma_{ij} \subset \Gamma$ be an interface such that $\lambda \neq 0$ on γ. Then we can obviously find a function $v \in H^1_0(\gamma)$ corresponding to λ such that

$$\int_{\gamma} v\lambda \, ds > 0. \tag{43}$$

We can now extend v to a function $V \in \tilde{V}_S$ which satisfies

$$\Delta V^i = 0, \quad \text{on } \Omega_i, \qquad V^i = v|_{\gamma}, \qquad V^i = 0, \quad \text{on } \partial\Omega_i/\gamma, \qquad V^j = 0, \quad \text{on } \Omega_j, \qquad j \neq i. \tag{44}$$

Using (37), (44), and (43), we have $b_S(V,\lambda) = \int_{\gamma} \left(V^i - V^j\right) \lambda \, ds = \int_{\gamma} v\lambda \, ds > 0$.

If $\mathcal{L}^{\gamma}_{h,\mathbf{k}}(v) \neq \emptyset$, then from (8) there exists $w \in \overset{0}{V}{}^{\gamma}_{h,\mathbf{k}}$ such that $\int_{\gamma}(w - v)\lambda \, ds = 0$. We can now use the extension operator $R^{\gamma}_{h,\mathbf{k}}$ developed in [4] to extend $w \in \overset{0}{V}{}^{\gamma}_{h,\mathbf{k}}$ to $v_h \in \tilde{V}_{h,\mathbf{k}}$ such that $v^i_h = w$ on γ and $v^j_h = 0$ for $j \neq i$. This, with $\mathcal{L}^{\gamma}_{h,\mathbf{k}}(v) \neq \emptyset$, then gives

$$b_S(v_h,\lambda) = \int_{\gamma} \left(v^i_h - v^j_h\right) \lambda \, ds = \int_{\gamma} w\lambda \, ds = \int_{\gamma} v\lambda \, ds > 0$$

by (43). This proves (42), and hence, the theorem. ∎

REMARK 5.2. Let us remark that the condition $u = 0$ on $\partial\Omega_D$ (or $u = g$ on $\partial\Omega_D$) could also be implemented by suitably modifying (37) to include appropriate boundary terms.

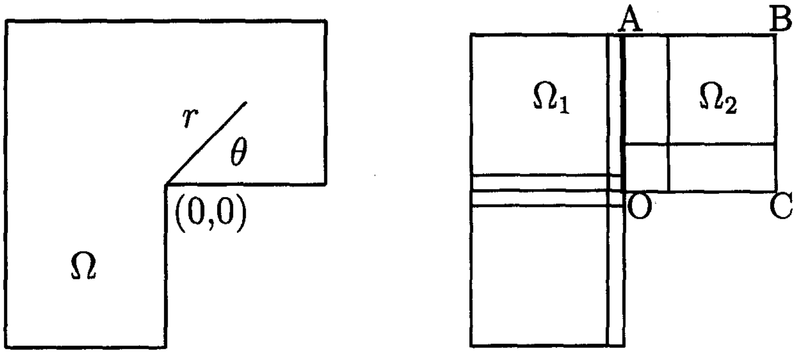
In [2], the above mixed method has been fully analyzed for M0 and an inf-sup condition has been derived. The argument is independent of the mesh choice and polynomial degree, and so carries over directly to nonquasiuniform meshes considered here. Hence, for the mixed method formulation of the nonconforming mortar methods, one can clearly bound the error in u and the error in the Lagrange multiplier λ. Numerical results presented in [5] validate the expected convergence behaviour for both u and λ for the model problem (1).

6. NUMERICAL RESULTS

In this section, we perform *hp* computations on geometric meshes for a variation of the model problem (1), implemented as a mixed method. We consider

$$-\operatorname{div}(\kappa\nabla u) = f, \quad \text{on } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega_D, \qquad \kappa\frac{\partial u}{\partial n} = g, \quad \text{on } \partial\Omega_N, \tag{45}$$

where $\kappa = \kappa_i$ is a smooth coefficient over each Ω_i, bounded by positive constants such that $a_i \leq \kappa_i \leq A_i$. Note that the entire analysis and constructions in this paper can be extended to (45) with minor modifications. If $\kappa = 1$, then (45) reduces to (1).



(a) L-shaped domain. (b) Partition and tensor product mesh for $m = n = 2$.

Figure 2.

We solve (45) on the L-shaped domain (Figure 2) where $\Omega = \Omega_1 \cup \Omega_2$ and $\kappa = \kappa_i$ on Ω_i. We let $\kappa_1 = 1$, $\kappa_2 = 2$ (different constants). This problem with discontinuous coefficients is of great interest in several applications. For instance, (45) models the case where one has two different materials, with conductivities κ_1 and κ_2. It can be verified that the dominant singularity at O behaves like r^α, where

$$\alpha = \frac{2}{\pi} \tan^{-1}\left(\sqrt{1 + 2\frac{\kappa_1}{\kappa_2}}\right). \tag{46}$$

Suppose we take Neumann boundary conditions $\partial\Omega_N = \partial\Omega$, with a Dirichlet condition imposed only at the point C. We choose f so that the exact solution is

$$u_1 = u|_{\Omega_1} = r^\alpha \left(\cos(\alpha\theta) + \tan\left(\frac{3\alpha\pi}{2}\right)\sin(\alpha\theta)\right) - c, \tag{47}$$

$$u_2 = u|_{\Omega_2} = c\left(r^\alpha \cos(\alpha\theta) - 1\right), \tag{48}$$

where $c = 1 + \tan(3\alpha\pi/2)\tan(\alpha\pi/2)$. Note that u given above satisfies (45). Also, across the interface OA, both the solution and the flux are continuous.

We impose appropriate Neumann conditions on $\partial\Omega$ to approximate problem (45) with exact solution (47),(48) by the mortar method M1 used along OA (the results for M0 being similar). We had presented the h-version results for (45) and the convergence rate was observed to be precisely $O(h^\alpha)$ for *uniform* meshes and $O(h^k)$ for *radical* meshes, where k was the degree of the polynomial [5]. Here we investigate the convergence rates for the p and hp versions.

For ease of implementation, we consider *tensor product meshes*, where Ω_2 is divided into n^2 rectangles and Ω_1 is divided into $2m^2$ rectangles (see Figure 2). For our experiments we take $m = n$, and along the x- and y-axes, take the grid points, $x_0 = 0$, $x_j = \sigma_i^{n-j}$ for $j = 1, \ldots, n$. Here, σ_i is the geometric ratio used on Ω_i. To make the method nonconforming we take $\sigma_1 = 0.17$ and $\sigma_2 = 0.13$. The conforming method (CF) is modelled by taking the geometric ratios to be the same in each domain (0.13 or 0.17 in each domain).

In Figure 3, we compare the performance of the nonconforming method with the conforming method using $n = 4$ layers. It clearly shows the characteristic 'S'-shaped p convergence curve with initial exponential convergence, followed by the flattened algebraic rate of $O(k^{-2\alpha})$ with α defined in (46) for $\kappa_1 = 1$, $\kappa_2 = 2$. Let us note that Theorem 3.1 suggested a possible loss of $O(k^{3/4})$ in the asymptotic rate. This is, however, not visible in the computations as the nonconforming and conforming slopes are the same. Figure 3 also demonstrates that the M1 method behaves as well as the conforming FEM.

Next, we plot the results of increasing the degree k for various n in Figure 4 using the M1 method. The hp version is then the lower envelope of these curves—by changing both n and k simultaneously, we remain in the exponential phase.

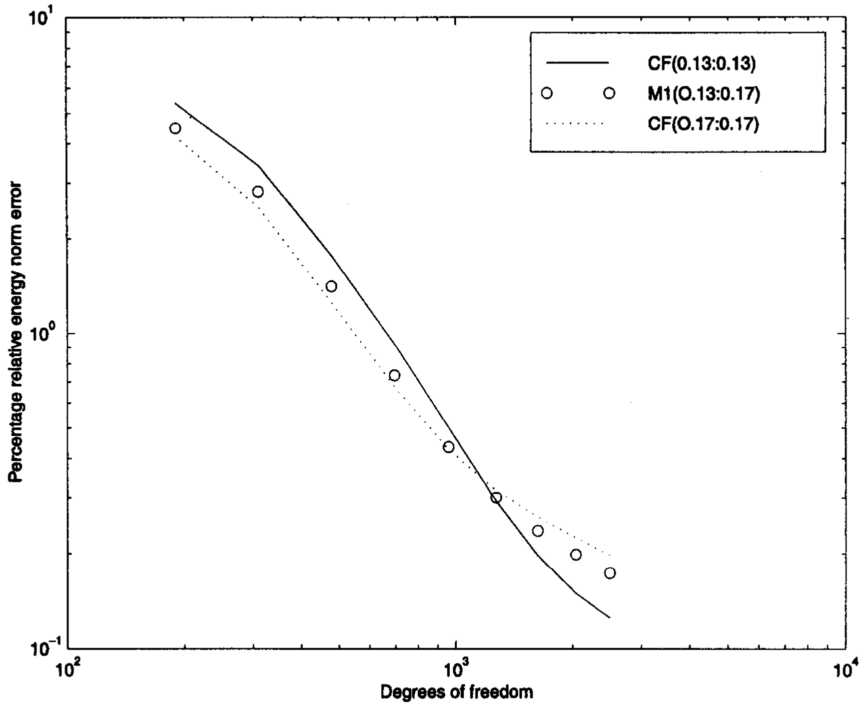


Figure 3. Comparison of M1 with the conforming FEM for p version over geometric mesh.

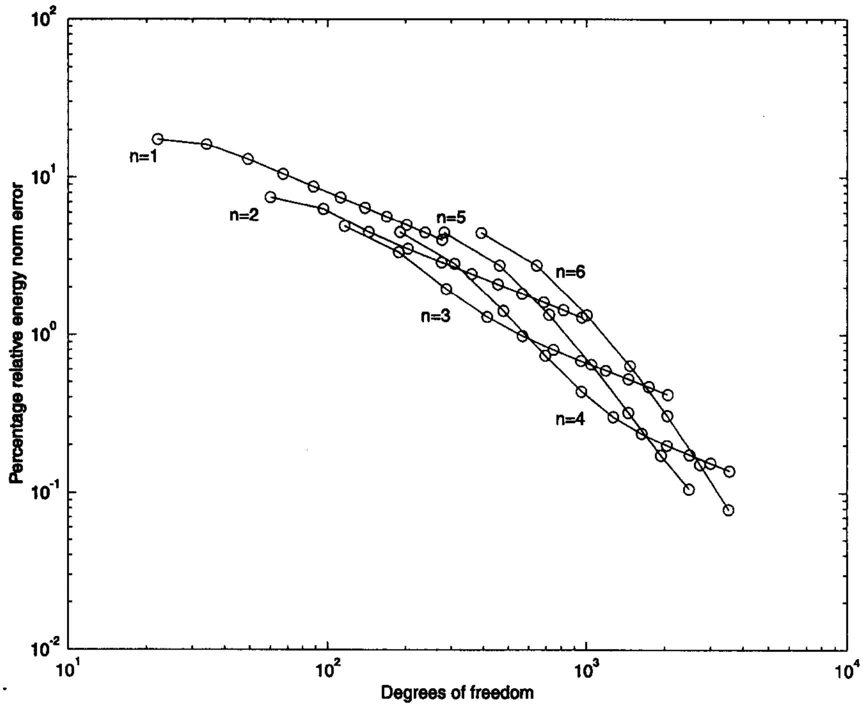


Figure 4. p version for geometric mesh using M1, $n = 1, \ldots, 6$, $\sigma_1 = 0.17$, $\sigma_2 = 0.13$.

Finally, in Figure 5, we plot the log of the relative energy norm error in u vs. $N^{1/4}$ (where N is the number of degrees of freedom), which gives a straight line. This shows that an exponential convergence rate of $Ce^{-\gamma N^{1/4}}$ is recovered when the nonconforming mortar finite-element method is used.

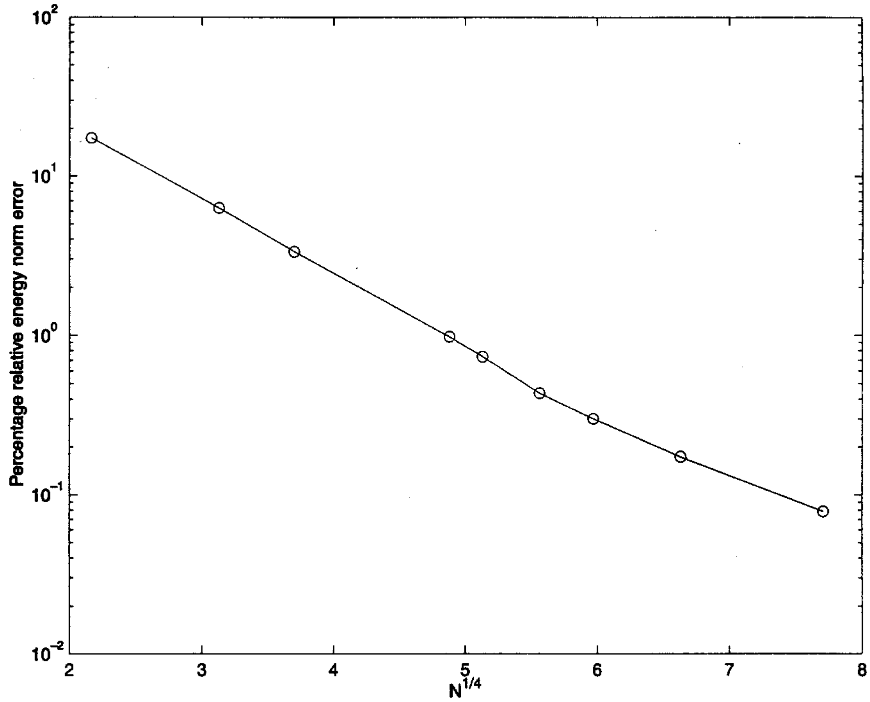


Figure 5. Exponential convergence for the *hp* version for M1 method.

7. ABSTRACT ERROR ESTIMATES IN 3-D

The extension of the definition of the mortar element methods M0 and M1 to three dimensions (for tensor product spaces) have been presented [5,8]. It can be seen that the M0 method cannot be easily defined for a general mesh of parallelograms (except for the linear case [7]). The advantage of M1 is clearly obvious since these extend easily to general meshes (and not just tensor product ones like M0). In order to prove abstract error estimates on the finite-element solution in 3-d, one has to use similar tools as in 2-d. Namely, we have to characterize the two main ingredients: the *stability measure* $\alpha_{h,\mathbf{k}}^{1/2+\epsilon}$ of the method as in (10),(11) and the bound on the *extension operator* $\beta_{h,\mathbf{k}}^{\epsilon,\gamma}$. We can then prove an abstract estimate in 3-d as in Theorem 2.1.

For the h version, an extension operator $R_{h,\mathbf{k}}^{\gamma}$ is defined and shown to satisfy (16),(17) with $\epsilon = 0$ and $\beta_{h,\mathbf{k}}^{\epsilon,\gamma} = \beta$ in Lemma 5.1 and Theorem 5.1 of [4]. These operators also exist for the p version [24] with $\epsilon = 0$. Hence, the only other thing left is to estimate the stability measure $\alpha_{h,\mathbf{k}}^{1/2+\epsilon}$. As before, for the M0 case, this measure is an upper bound on the norm of a unique projection operator defined by (7) and (8). We will estimate the norm of this projection operator and this will then help us to estimate the dependence of $\alpha_{h,\mathbf{k}}^{1/2+\epsilon}$ on k which can then be used in the abstract error estimate. Our proof is for any tensor product interface mesh defined on Γ_{ij}. We will estimate the bounds of this operator in $L_2(I \times I)$ and $H_0^1(I \times I)$, which will enable us to then estimate $\alpha_{h,\mathbf{k}}^{1/2+\epsilon}$ by interpolation. We begin with a general tensor product result.

7.1. A Tensor Product Result

Let I be an interval and $Q = I \times I$. Let V and S be *tensor product spaces* on Q of the form

$$V = \mathrm{Span}\{\mu(x)\rho(y) \text{ where } \mu(x) \in V_x,\ \rho(y) \in V_y\} = V_x \otimes V_y,$$
$$S = \mathrm{Span}\{\chi(x)\psi(y) \text{ where } \chi(x) \in S_x,\ \psi(y) \in S_y\} = S_x \otimes S_y,$$

where $V_x(I)$, $V_y(I)$, $S_x(I)$, $S_y(I)$ are spaces each defined on I.

We assume V, S are such that we can define the projection $\Pi : L_2(Q) \to V(Q)$ as follows: for $u(x,y) \in L_2(Q)$, $\Pi u \in V(Q)$ satisfies, for all $\Phi \in S(Q)$,

$$\int_I \int_I \Pi u \Phi \, dx \, dy = \int_I \int_I u \Phi \, dx \, dy. \tag{49}$$

Moreover, the following 1-d projections are also defined. For any function $u(x)$, $\Pi_x u \in V_x(I)$ satisfies

$$\int_I \Pi_x u \chi \, dx = \int_I u \chi \, dx, \qquad \forall \chi \in S_x(I). \tag{50}$$

Similarly, for any function $u(y)$, $\Pi_y u \in V_y(I)$ satisfies

$$\int_I \Pi_y u \psi \, dy = \int_I u \psi \, dy, \qquad \forall \psi \in S_y(I). \tag{51}$$

We will be using the notation $\Pi_y \Pi_x u(x,y)$ in the following theorem which we define as follows. Consider the function $u(x,y)$. We first fix $y = \bar{y}$, which gives a function $u(x,\bar{y})$. This is a function of x alone, and hence, we can apply Π_x to this function and define

$$\xi(x,\bar{y}) = \Pi_x u(x,\bar{y}). \tag{52}$$

This can be repeated for each $\bar{y}$ to get a new function, $(\Pi_x u)(x,y)$. Now we fix $x = \bar{x}$ to give a function $(\Pi_x u)(\bar{x},y)$ which depends only on y. Hence, applying Π_y to $(\Pi_x u)(\bar{x},y)$, we get

$$\sigma(\bar{x},y) = \Pi_y(\Pi_x u)(\bar{x},y) = \Pi_y(\xi(\bar{x},y)). \tag{53}$$

Repeating this for each $\bar{x}$ we finally get a new function which we denote by $\Pi_y \Pi_x u(x,y)$.

Note that $\Pi_x \Pi_y u(x,y)$ can be defined analogously. The main result of this section is the following.

LEMMA 7.1. *For any $u \in L_2(I)$,*

$$\Pi u = \Pi_x \Pi_y u = \Pi_y \Pi_x u. \tag{54}$$

PROOF. Πu is the unique element in V which satisfies (49). In order to establish (54), we will prove that $\Pi_x \Pi_y u$ also satisfies (49). Let us fix $y = \bar{y}$. For each $\bar{y}$, we know that $u(.,\bar{y}), \Pi_x u(.,\bar{y})$ are functions of x alone. The definition of $\Pi_x u(x,y)$ from (50) gives

$$\int_I \left((\Pi_x u)(x,\bar{y}) - u(x,\bar{y}) \right) \chi(x) \, dx = 0. \tag{55}$$

Note that both $\int_I \Pi_x u(x,y)\chi(x)\,dx$ and $\int_I u(x,y)\chi(x)\,dx$ are functions of y. Hence, we can apply Π_y to both sides of (55) (with $\bar{y}$ replaced with y) and then multiply by the function $\psi(y)$ on both sides of the resulting equation and integrate for all y. This then gives

$$\int_I \Pi_y \left(\int_I \Pi_x u(x,y)\chi(x)\,dx \right) \psi(y)\,dy = \int_I \Pi_y \left(\int_I u(x,y)\chi(x)\,dx \right) \psi(y)\,dy. \tag{56}$$

Consider the left-hand side of (56). Using the definition of Π_y from (51), we get

$$\int_I \Pi_y \left(\int_I \Pi_x u(x,y)\chi(x)\,dx \right) \psi(y)\,dy = \int_I \int_I \Pi_x u(x,y)\psi(y)\,dy\,\chi(x)\,dx,$$

to which one can now apply the definition of Π_y again to give

$$\int_I \Pi_y \left(\int_I \Pi_x u(x,y)\chi(x)\,dx \right) \psi(y)\,dy = \int_I \int_I \Pi_y \left(\Pi_x u(x,y) \right) \psi(y)\chi(x)\,dx\,dy. \tag{57}$$

Now let us consider the right-hand side of (56). We first use the definition of Π_y to give

$$\int_I \Pi_y \left(\int_I u(x,y)\chi(x)\,dx \right) \psi(y)\,dy = \int_I \int_I u(x,y)\psi(y)\chi(x)\,dx\,dy. \tag{58}$$

From (56)–(58) we have

$$\int_I \int_I \Pi_y \left(\Pi_x u(x,y) \right) \psi(y)\chi(x)\,dx\,dy = \int_I \int_I u(x,y)\psi(y)\chi(x)\,dx\,dy. \tag{59}$$

Comparing equations (49) and (59), we then see that $\Pi u = \Pi_y \Pi_x u$, which is one part of (54). The other part is proved in the same way. ∎

7.2. Estimate in $L_2(Q)$

The projection operator Π in (49) satisfies the following L_2 estimate.

THEOREM 7.1. *For $u \in L_2(Q)$, $\|\Pi u\|_{0,Q} \le Ck\|u\|_{0,Q}$.*

PROOF. We first recall the one-dimensional estimates obtained in Theorem 3.1 in [4] for the projection operator

$$\|\Pi_x v\|_{0,I} \le Ck^{1/2}\|v\|_{0,I}, \qquad \text{for any } v(x) \in L_2(I), \tag{60}$$

$$\|\Pi_y v\|_{0,I} \le Ck^{1/2}\|v\|_{0,I}, \qquad \text{for any } v(y) \in L_2(I). \tag{61}$$

Let us now estimate the L_2 norm over I of the function $\xi(x,\bar{y})$ defined by (52). We have, by (60),

$$\|\xi(x,\bar{y})\|_0^2 = \int_I (\Pi_x u(x,\bar{y}))^2 \, dx = \|\Pi_x u(\,.\,,\bar{y})\|_{0,I}^2 \le Ck\,\|u(\,.\,,\bar{y})\|_{0,I}^2 = Ck\int_I (u(x,\bar{y}))^2 \, dx.$$

Varying $\bar{y}$ for all y and integrating in the y direction gives

$$\int_I\int_I ((\Pi_x u)(x,y))^2 \, dx\, dy \le Ck\int_I\int_I (u(x,y))^2 \, dx\, dy. \tag{62}$$

Similarly, using (53) and (61) it can be shown that $\|\sigma(\bar{x},y)\|_{0,I}^2 \le k\|\Pi_x u\|_{0,I}^2$. This gives

$$\int_I (\Pi_y(\Pi_x u)(\bar{x},y))^2 \, dy \le k\int_I ((\Pi_x u)(\bar{x},y))^2 \, dy.$$

Now varying $\bar{x}$ for all x and integrating in the x direction, the above equation then becomes

$$\int_I\int_I (\Pi_y(\Pi_x u)(x,y))^2 \, dy\, dx \le k\int_I\int_I ((\Pi_x u)(x,y))^2 \, dy\, dx. \tag{63}$$

From (62) and (63), we get the result. ∎

7.3. Estimate in $H_0^1(Q)$

We will now estimate the projection operator Π in $H_0^1(Q)$. For this we will need the following lemma.

LEMMA 7.2. *For $u \in H_0^1(Q)$,*

$$\frac{\partial}{\partial x}(\Pi_y u) = \Pi_y \frac{\partial u}{\partial x}, \tag{64}$$

$$\frac{\partial}{\partial y}(\Pi_x u) = \Pi_x \frac{\partial u}{\partial y}. \tag{65}$$

PROOF. For any $\psi \in S_y(I)$, since ψ is a function of y only,

$$\int_I \frac{\partial}{\partial x}(\Pi_y u)\psi \, dy = \frac{\partial}{\partial x}\int_I \Pi_y u\psi \, dy = \frac{\partial}{\partial x}\int_I u\psi \, dy, \tag{66}$$

where we have used (51) for the last part. Now using again the fact that ψ depends only on y gives

$$\frac{\partial}{\partial x}\int_I u\psi \, dy = \int_I \left(\frac{\partial u}{\partial x}\right)\psi \, dy = \int_I \Pi_y\left(\frac{\partial u}{\partial x}\right)\psi \, dy, \tag{67}$$

where the last equality is from (51). Equation (64) follows by combining (66) and (67). Equation (65) is proved in a similar fashion. ∎

THEOREM 7.2. *For all $u \in H_0^1(Q)$,*

$$\|\Pi u\|_{1,Q} \le C k^{3/2} \|u\|_{1,Q}. \tag{68}$$

PROOF. From Lemma 7.1 and (64),

$$\left\| \frac{\partial}{\partial x}(\Pi u) \right\|_{0,I} = \left\| \frac{\partial}{\partial x} (\Pi_y \Pi_x u) \right\|_{0,I} = \left\| \Pi_y \left(\frac{\partial}{\partial x} (\Pi_x u) \right) \right\|_{0,I}. \tag{69}$$

Using (61) and Theorem 3.1 of [4], we also have

$$\left\| \Pi_y \left(\frac{\partial}{\partial x} (\Pi_x u) \right) \right\|_{0,I} \le C k^{1/2} \left\| \frac{\partial}{\partial x} (\Pi_x u) \right\|_{0,I} \le C k^{1/2} \left\| (\Pi_x u)' \right\|_{0,I} \le C k^{3/2} \|u\|_{1,I}.$$

Combining the above equation with (69) proves

$$\left\| \frac{\partial}{\partial x} (\Pi u) \right\|_{0,I} \le C k^{3/2} \|u\|_{1,I}. \tag{70}$$

Similarly, we can also prove $\|\frac{\partial}{\partial y} (\Pi u)\|_{0,I} \le C k^{3/2} \|u\|_{1,I}$. Estimate (68) then follows from this and (70). ∎

Theorems 7.1 and 7.2 can now be used to obtain the following abstract error estimate in three-dimensions for any tensor product interface mesh.

THEOREM 7.3. *The stability measure for the mortar finite-element methods M0 and M1 satisfies*

$$\alpha_{h,\mathbf{k}}^{1/2+\epsilon} \le C k^{5/4+\epsilon}. \tag{71}$$

PROOF. For the M0 method, since the constants $\alpha_{h,\mathbf{k}}^{0,\gamma}$ and $\alpha_{h,\mathbf{k}}^{1,\gamma}$ provide upper bounds for the projection operator Π, we have by Theorems 7.1 and 7.2 $\alpha_{h,\mathbf{k}}^{0,\gamma} \le O(k)$ and $\alpha_{h,\mathbf{k}}^{1,\gamma} \le O(k^{3/2})$. Using (14), we can then obtain $\tilde{\alpha}_{h,\mathbf{k}}^{1/2+\epsilon,\gamma} \le C k^{5/4+\epsilon}$. Definition (11) of the stability measure and (15) then give $\alpha_{h,\mathbf{k}}^{1/2+\epsilon} = \alpha_{h,\mathbf{k}}^{1/2+\epsilon,\gamma} \le \tilde{\alpha}_{h,\mathbf{k}}^{1/2+\epsilon,\gamma}$. This proves (71) for M0. Estimate (71) follows for method M1 by using (6) and (8). ∎

Although we do not have theoretical estimates for nontensor product type spaces, it has been shown that we can still computationally test the invertibility of various mortar methods for nontensor product meshes [8]. Let us remark that we could also computationally estimate the stability measures for various h, p, and hp methods as in the 2-d case. The only difference is that we now must use a truncated double sum instead of (25) and (34). For the k-dependence of mortar methods in 3-d, we believe that such a computational study would be a good alternative to trying to mathematically establish stability (which is difficult in the absence of a tensor product structure).

REFERENCES

1. C. Bernardi, Y. Maday and A.T. Patera, Domain decomposition by the mortar element method, In *Asymptotic and Numerical Methods for PDEs with Critical Parameters*, Volume 384, (Edited by H.G. Kaper and M. Garbey), NATO Adv. Sci. Inst. Ser. C Math. Phs. Sci., pp. 269–286, Kluwer, (1993).
2. F. Ben Belgacem, The mortar finite element method with Lagrange multipliers, *Numer. Math.* **84**, 173–197, (1999).
3. M. Cassarin and O.B. Widlund, A hierarchical preconditioner for the mortar finite element method, *ETNA, Electron. Trans. Numer. Anal.* **4**, 75–88, (1996).
4. P. Seshaiyer and M. Suri, Uniform *hp* convergence results for the mortar finite element method, *Math. Comp.* **69**, 521–546, (2000).
5. P. Seshaiyer and M. Suri, *hp* submeshing via non-conforming finite element methods, *Comp. Meth. Appl. Mech. Engrg.* **189**, 1011–1030, (2000).

6. F. Ben Belgacem, P. Seshaiyer and M. Suri, Optimal convergence rates of *hp* mortar finite element methods for second-order elliptic problems, *RAIRO Math. Mod. Numer. Anal.* **34**, 591–608, (2000).
7. F. Ben Belgacem and Y. Maday, The mortar element method for three dimensional finite elements, *RAIRO Math. Mod. Numer. Anal.* **31**, 289–302, (1997).
8. P. Seshaiyer, Non-conforming *hp* finite element methods, Ph.D. Thesis, Department of Mathematics and Statistics, University of Maryland, Baltimore County, (1998).
9. P.A. Raviart and J.M. Thomas, Primal hybrid finite element methods for 2^{nd} order elliptic equations, *Math. Comp.* **31**, 391–396, (1977).
10. M. Dorr, On the discretization of inter-domain coupling in elliptic boundary-value problems via the *p*-version of the finite element method, In *Domain Decomposition Methods*, (Edited by T.F. Chan, R. Glowinski, J. Periaux and O.B. Widlund), pp. 17–37, SIAM, (1989).
11. H. Swann, On the use of Lagrange multipliers in domain decomposition for solving elliptic problems, *Math. Comp.* **60**, 49–78, (1993).
12. R.E. Ewing, R.D. Lazarov, T. Lin and Y. Lin, Mortar finite volume element methods of second order elliptic problems, *East-West J. Numer. Math.* **8** (2), 93–110, (2000).
13. J. Gopalakrishnan and J.E. Pasciak, Multigrid for the mortar finite element method, *SIAM J. Numer. Anal.* **37**, 1029–1052, (2000).
14. D. Braess, M. Dryja and W. Hackbusch, Multigrid method for nonconforming finite element discretization with application to non-matching grids, *Comput.* **63**, 1–25, (1999).
15. P.G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, (1978).
16. I. Babuška and M. Suri, The optimal convergence rate of the *p* version of the finite element method, *SIAM J. Numer. Anal.* **24**, 750–776, (1987).
17. I. Babuška and M. Suri, The *hp* version of the finite element method with quasiuniform meshes, *RAIRO Math. Mod. Numer. Anal.* **21**, 199–238, (1987).
18. O.B. Widlund, An extension theorem for finite element spaces with three applications, In *Numerical Techniques in Continuum Mechanics, Proceedings of the GAMM Seminar*, (Edited by W. Hackbush and K. Witdr), Kiel, 1986.
19. J. Xu and J. Zou, Some non-overlapping domain decomposition methods, *SIAM Review* **40** (4), 857–914, (1998).
20. P. Seshaiyer and M. Suri, Convergence results for non-conforming *hp* methods: The mortar finite element method, *Contemp. Math.* **218**, 453–459, (1998).
21. F. Ben Belgacem and Y. Maday, Non-conforming spectral element methodology tuned to parallel implementation, *Comp. Meth. Appl. Mech. Engrg.* **116**, 59–67, (1994).
22. J.T. Oden, A. Patra and Y. Fend, Domain decomposition for adaptive *hp* finite element methods, *Contemp. Math.* **180**, 295–301, (1994).
23. L. Demkowicz, J.T. Oden, W. Rachowicz and O. Hardy, Toward a universal *h-p* adaptive element strategy, Part I: Constrained approximation and data structure, *Comp. Meth. Appl. Mech. Engrg.* **77**, 79–113, (1989).
24. F. Ben Belgacem, Discretisations 3D non conformes pour la méthode de decomposition de domaine des elément avec joints: Analyze mathématique et mise en œvre pour le probleme de Poisson, Ph.D. Thesis, Université Pierre et Marie Curie, Paris, (January 1993).